Substituting Eq. (4) into Eq. (3), one obtains Martino's equation for the slip regime:

$$X = [X_{con} + (M/Re^{1/2})(X_{fm})]/[1 + (M/Re^{1/2})] \quad (5)$$

Figures 1–4 show how Martino's equation [Eq. (5)] compares with experimental data. The free molecule aerodynamic coefficients used in Eq. (5) were obtained from Blick.[5] Figure 1 shows a comparison between Martino's equation and the method of Lukasiewicz et al.,[6] which is essentially a combination of Bertram's[8] viscous interaction skin friction correction and Li and Nagamatsu's[9] induced pressure correction modified by the Mangler transformation.

In the transition regime, the Knudsen number given by Eq. (4) is probably not the best one to use. If one assumes that the characteristic length is proportional to the shock-detachment thickness and the mean free path evaluated behind the shock, then the Knudsen number can be defined as

$$K_n = \beta\lambda_s/\Delta = \beta\rho_s\lambda_s/\rho_1 D \quad (6)$$

If the stagnation temperature is low, then $\lambda\rho$ is insensitive to temperature, and Eq. (6) would reduce to

$$K_n = \beta\lambda_1/D \quad (7)$$

$\beta$, defined here to be the "Martino number," is simply a numerical factor that fits the Martino equation [Eq. (3)] as close as possible to the experimental data. Experimental drag data from Bloxsom and Rhodes[10] were correlated by Martino's equation [Eq. (3)], along with Eq. (6), in Fig. 5. Each shape in Fig. 5 had a different Martino number. It was found that the Martino numbers could be correlated by the following equation:

$$\beta = \exp[3.36 - 4.26\,(C_{D_{con}})/(C_{D fm})] \quad (8)$$

It is not known at this time whether the drag Martino number given by Eq. (8) is applicable to other aerodynamic coefficients. If it is, then one can simply substitute the continuum to free molecule ratio of the coefficient into Eq. (8) in place of the drag coefficient ratio. However, further experimental data on coefficients (other than drag coefficients) will have to be obtained before the validity of his method can be checked.

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# Flapping Propulsion Wake Analysis


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A WING, oscillating normal to its direction of flight, experiences a propulsive force that can be evaluated from its wake characteristics. A classic wake formula is adapted to relate thrust force to the flapping frequency and forward speed, disclosing important distinctive features of unsteady propulsion. It is shown that a portion of the thrust remains finite as forward speed tends to zero. Aside from its interest for natural and low-speed flight, the result is applicable in such domains as underwater propulsion, by reason of the fact that the thrust mechanism is independent of normal (i.e., lift) force.

Unsteady wing propulsion, once termed Katzmayr effect, furnishes a specific physical representation as well as a useful terminology; the propulsor will be henceforth referred to as a wing, although the discussion applies equally well to various vortex-shedding configurations. A variety of wing and flap oscillation modes are known which approach ideal mechanical efficiency, and, in all cases, a thick wake is formed which consists of the flow region bounded by two staggered rows of oppositely directed vorticity. Except for the fact that the vortex sense is reversed (hence, also the direction of the proper vortex motion), the vortex pattern is identical to the vortex street of Bénard and Kármán. The thrust force is thus given directly by a slight modification of Kármán's formula:

$$T = \rho V\Gamma\frac{h}{l} + \rho\frac{\Gamma^2}{l}\left(\frac{h/l}{2^{1/2}} - \frac{1}{2\pi}\right) \quad (1)$$

where $\Gamma$ is the vortex strength, $V$ is the forward speed, and $h/l$ is the ratio of street width to streamwise vortex spacing in either row (see, e.g., Ref. 1). The latter ratio being a known constant, it is convenient to regard the parameter $l$ as a measure of wing oscillation amplitude.

The circulation $\Gamma$ can be replaced by the cyclic frequency $f$ of wing oscillation by noting that this is the ratio of speed $V + u_T$ of vortex relative to wing divided by spacing $l$:

$$f = (V + u_T)/l$$

The motion $u_T$ of the vortex relative to the freestream is known from vortex dynamics as

$$u_T = \frac{1}{2(2)^{1/2}}\frac{\Gamma}{l}$$

so that

$$f = \frac{V}{l}\left\{1 + \frac{1}{2(2)^{1/2}}\frac{\Gamma}{lV}\right\} \quad (2)$$

Substitution for $\Gamma$ in (1) gives the thrust dependence on frequency and forward speed:

$$T = \rho V\frac{h}{l}\,(2)^{1/2}l(fl - V) + \rho l\cdot 8\left(\frac{h/l}{2^{1/2}} - \frac{1}{2\pi}\right)(fl - V)^2 \quad (1')$$

Although both terms on the right side of (1′) include the familiar velocity-squared terms, the second term is also seen to contain a contribution to the thrust, which is independent of forward speed. This feature is the principal reason for the importance of oscillating wings in low-speed flight technology. Thrust being independent of lift, moreover, it ap-


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pears that customary unfavorable low-speed induced drag limitations can be ameliorated by these means.

With regard to flapping frequency, it is clear from (1′) that any value in excess of $V/l$ leads to positive thrust, numerical coefficients of both terms on the right side being positive. At low forward speed, therefore, $f$ may be arbitrarily small, the limit of static thrust ($V = 0$) corresponding to thrust proportional to square of frequency. It is also found that no value of flapping frequency below $V/l$ leads to positive thrust, regardless of speed $V$.

The contrast with steady-flow force phenomena is further clarified by interpreting separately the two terms on the right side of (1). The first of these, despite the striking formal resemblance to the Kutta-Joukowsky steady lift formula, exhibits a completely different character in the present case. Instead of circulation proportional to speed, as in the steady airfoil theory, Eq. (2) shows that $\Gamma$ is proportional to frequency in the limit of low speeds, so that the thrust contribution represented by the term in question vanishes only as the first power of $V$. At higher speeds this term approaches the $V$-square dependence as in steady flow. The second term, containing a part independent of speed $V$, therefore dominates the force effects at low speeds. This term has no counterpart in steady lifting flows.

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# Transcendental Approximation for Laminar Boundary Layers

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In the fields of boundary-layer theory and unsteady thermal conduction, there is considerable use for profile methods.[1,2] These methods are applied to the solution of problems where a function is known to exist, usually in a semi-infinite space, which is known to have its greatest variation in magnitude close to one of the boundaries. Usually it is known that the function is monotonic also. The profile methods that are commonly used at present are approximations in two ways: first the outer boundary condition is brought from infinity to a finite distance from the inner surface, and then the profile is approximated by the use of a polynomial. The profile is then required to satisfy some integral condition. Since the region over which the function is approximated is finite, Weierstrass' theorem can be invoked to protect the use of the polynomial. However, it would seem that this theorem could be too strong for what is required. Furthermore, the degree of closeness of a low-order polynomial to the true function is very small, and this can adversely affect computations of hydrodynamic stability.

There seems to be some merit in exploring the possibility of finding simple, transcendental, approximate profiles that could be used in such integral methods. In order to obtain some hint of a suitable profile, it is useful to examine available numerical solutions for a typical problem, the incompressible boundary-layer flow over a wedge, i.e., the Falkner-Skan-Hartree problem. The exponential function is an elementary transcendent, and since the profiles resemble "decay" curves it is natural to examine the local logarithmic decrement. This is displayed in Fig. 1, in which $\eta$ is the dimensionless space similarity parameter, $f$ is the dimensionless stream function, and $\beta$ is the wedge parameter; $\beta > 0$ corresponds to accelerated flows. From the figure it is seen that there is a considerable indication of linearity of the local logarithmic decrement; the profile converges with increasing strength to its outer boundary condition. The linearity is particularly noticeable for $\beta \geq 0$ and remains a reasonable approximation for $\beta < 0$. Accordingly, a profile function can be written

$$\text{pro}\eta = \exp[-\exp(a + b\eta)]/\exp[-\exp a]$$

such that $\text{pro}\eta = 1$ at $\eta = 0$, $\text{pro}\eta \to 0$ as $\eta \to \infty$, and where the function is monotonic, provided that $a$ and $b$ are real and $b$ is positive. For severe cases with $\beta < 0$, it might be possible to use inner and outer expansions of this linear type, or, alternatively, to use an argument $(a + b\eta + c\eta^2)$.

Computations for the wedge-flow laminar boundary layer have been performed using the profile function already given to obtain both velocity profiles and minimum critical Reynolds numbers as functions of the wedge parameter $\beta$. The profile function contains two parameters that are functions of $\beta$, and, consequently, it is necessary to use two simultaneous

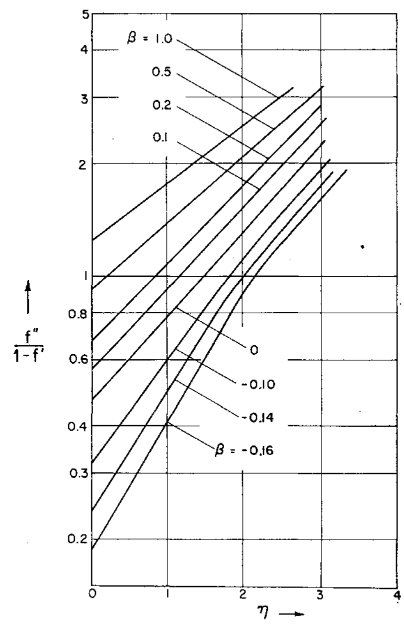


**Fig. 1 Local logarithmic decrement in the wedge-flow boundary layer.**

equations (the integral momentum and energy equations) to determine them. It was found that the integrals involved could be reduced quite readily to the exponential integral, which is extensively tabulated. Comparison of this profile function with the exact solution for both wall shear and minimum critical Reynolds number is excellent, except for values of $\beta$ close to that for separation. For values of $\beta > -0.08$, the shear stress is within 0.3% of Hartree's values, and the minimum critical Reynolds number is within 3% of Tetervin's values,[3] which are based on Hartree's profiles. Details of the computations are available elsewhere.[4]

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